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knowledge is power

THIRUTHANGAL NADAR COLLEGE

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Selavayal, Chennai-51.

A Self-Financing Co-educational College of Arts & Science

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NAME OF THE DEPARTMENT : MATHEMATICS

SUBJECT : MATHEMATICS - II (ALLIED)

TOPIC : LAPLACE TRANSFORM

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LAPLACE TRANSFORMS

Introduction

- Transformation in mathematics deals with the conversion of one function to another function that may not be in the same domain.

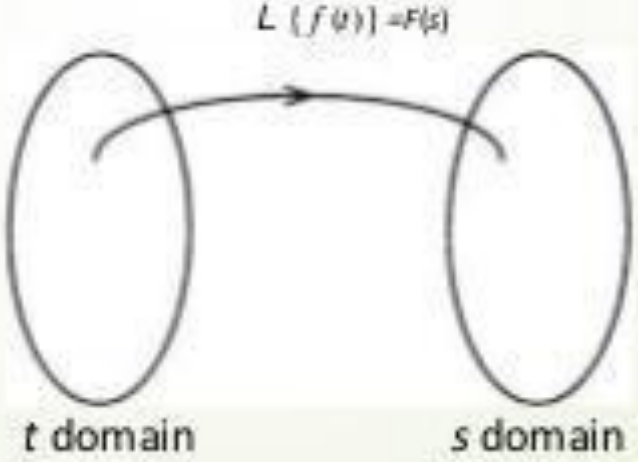
- Laplace transform is a powerful transformation tool, which literally transforms the original differential equation into an elementary algebraic expression. This latter can then simply be transformed once again, into the solution of the original problem.

- This transform is named after the mathematician and renowned astronomer Pierre Simon Laplace who lived in France.

Definition of Laplace Transform

Suppose that, f is a real or complex valued function of the variable $t > 0$ and s is a real or complex parameter. We define the Laplace transform of f as

$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$$



Existence of Laplace Transform

- f(t) must be piecewise continuous which means that it must be single valued but can have a finite number of finite isolated discontinuities for $t > 0$.
- $f(t)$ must be exponential order which means that $f(t)$ must remain less than Me^{-at} as t approaches ∞ where M is a positive constant and a is a real positive number.

Elementary Properties of Laplace Transformation

- **Linear Property :** If c_1 and c_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transforms $f_1(s)$ and $f_2(s)$, then L $\{c_1F_1(t)+c_2F_2(t)\}=c_1f_1(s)+c_2f_2(s)$
- **First translation or shifting property :** If L{F(t)}=f(s), then L$\{e^{at}F(t)\}$=f(s-a)
- **Second translation or shifting property :** If L{F(t)}=f(s) and G(t)= $\begin{cases} F(t-a); t > a \\ 0 \quad ; t < a \end{cases}$

  then L{G(t)} = $e^{-as}f(s)$
- **Laplace transformation of derivatives:** If L{F(t)} = f(s), then L(F'(t)) = s f(s) - F(0)
- **Laplace transformation of integral:** If L{F(t)} =f(s), then L$\{\int_0^t F(u)du\}$ =$\frac{f(s)}{s}$
- **Multiplication by t^n:** If L{F(t)}=f(s), then L$\{t^n F(t)\}=(-1)^n f^{(n)}(s)$
- **Division by t:** If L{F(t)}=f(s), then L$\{\frac{F(t)}{t}\}=\int_s^{\infty} f(u)\,du$ provided $\lim_{t\to 0} F(t)/t$ exists.

Laplace Transform Of Some Basic Function

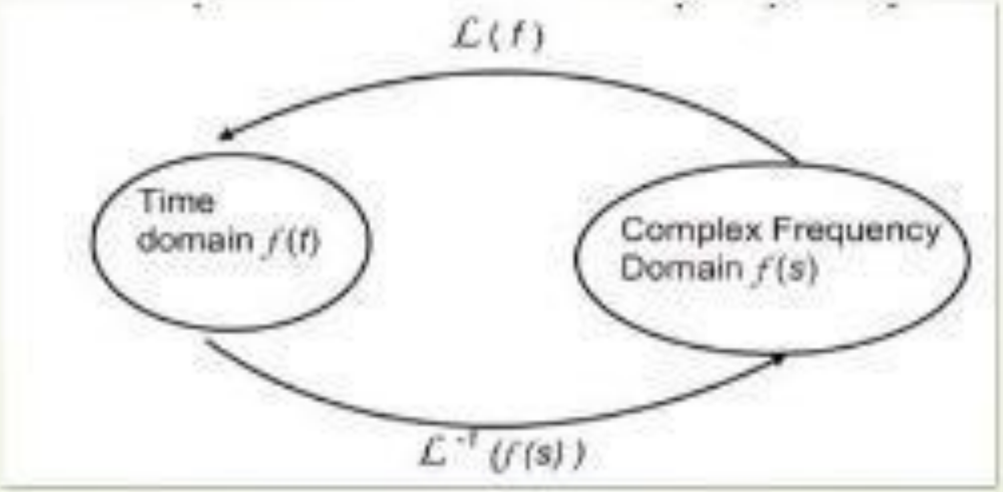
$f(t)$	$\mathcal{L}(f(t))$	$f(t)$	$\mathcal{L}(f(t))$
1	$\frac{1}{s}$	$\cos^2 kt$	$\frac{s^2+2k^2}{s(s^2+4k^2)}$
t	$\frac{1}{s^2}$	e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh kt$	$\frac{k}{s^2-k^2}$
$t^{-1/2}$	$\sqrt{\frac{\pi}{s}}$	$\cosh kt$	$\frac{s}{s^2-k^2}$
$t^{1/2}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	$\sinh^2 kt$	$\frac{2k^2}{s(s^2-4k^2)}$
$\sin kt$	$\frac{k}{s^2+k^2}$	$\cosh^2 kt$	$\frac{s^2-2k^2}{s(s^2-4k^2)}$
$\cos kt$	$\frac{s}{s^2+k^2}$	te^{at}	$\frac{1}{(s-a)^2}$
$\sin^2 kt$	$\frac{2k^2}{s(s^2+4k^2)}$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$ n a positive integer

Inverse Laplace Transform

Definition of Inverse Laplace Transform

In order to apply the Laplace transform to physical problems, it is necessary to invoke the inverse transform. If L{f(t)}= F(s), then the inverse Laplace Transform is denoted by

$$L^{-1}\{F(s)\} = f(t), \quad t \geq 0$$



Elementary Properties of Inverse Laplace transform

- **Linearity property :** If c_1 and c_2 are any constant while $f_1(s)$ and $f_2(s)$ are the Laplace transform of $F_1(t)$ And $F_2(t)$ respectively, then $L^{-1}\{c_1f_1(s)+c_2f_2(s)\}=c_1F_1(t)+c_2F_2(t)$
- **First translation property :** If $L^{-1}\{f(s)\}$=F(t), then $L^{-1}\{f(s-a)\}= e^{at}F(t)$
- **Inverse Laplace transformation of integral :** If $L^{-1}\{f(s)\}$=F(t), then $L^{-1}\{\int_s^{\infty} F(u)du\}=\frac{F(t)}{t}$
- **Inverse Laplace transformation of derivatives :** If $L^{-1}\{f(s)\}$=F(t), in that case $L^{-1}\{f^{(n)}(s)\}=L^{-1}\{\frac{d^n}{ds^n}f(s)\}= (-1)^n t^n F(t)$
- **Multiplication by s :** If $L^{-1}\{f(s)\}$=F(t), f(0)=0 it follows that $L^{-1}\{sf(s)\}=F'(t)$
- **Division by s :** If $L^{-1}\{f(s)\}$=F(t), it follows that $L^{-1}\{\frac{f(s)}{s}\}=\int_0^t F(u)du$
- **Inverse Laplace transformation of product of two functions :** If $L^{-1}\{f(s)\}$=F(t) and $L^{-1}\{g(s)\}$=G(t), then $L^{-1}\{f(s)*g(s)\}=\int_0^t F(u)G(t-u)du$ Which is known as convolution of F and G

Inverse Laplace Transform Of Some Basic Function

f (s)	$L^{-1}\{f(s)\}$ =F(t)
t^n	$\frac{t^n}{n!}$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^2+a^2}$	$\frac{\text{Sin at}}{a}$
1	$\delta(t)$
e^{-as}	$\delta(t-a)$
$\frac{1}{s^2-a^2}$	$\frac{\text{sinh at}}{a}$
$\frac{1}{s-x}; s > x$	e^{xt}
$\frac{1}{(s-a)^2}$	$t\,e^{at}$
$\frac{x}{s^2+x^2}$	Sin xt
$\frac{s}{s^2+x^2}; s > 0$	Cos xt

Application of Laplace Transform

Solving Ordinary Differential Equation

Problem:

$Y'' + aY' + bY = G(t)$ subject to the initial conditions $Y(0) = A$, $Y'(0) = B$ where a, b, A, B are constants.

Solution:

- Laplace transform of $Y(t)$ be $y(s)$, or, more concisely, y.
- Then solve for y in terms of s.
- Take the inverse transform, we obtain the desired solution Y.

Solving Partial Differential Equation

Problem: Solve $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$

with the boundary conditions U(x, 0) = 3 sin 2πx, U(0, t) = 0 and U(1, t) = 0

where 0 < x < 1, t > 0.

Solution:

- Taking Laplace transform of both sides with respect to t,

  $$su - U(x,0) = \frac{d^2u}{dx^2}$$

- Substituting in the value of U(x, 0) and rearranging, we get

  $$1)\quad \frac{d^2u}{dx^2} - su = -3\sin 2\pi x$$

  where u = u(x, s) = L[U(x, t)].

- The general solution of (1) is $2)\quad u = c_1 e^{\sqrt{s}\,x} + c_1 e^{-\sqrt{s}\,x} + \frac{3}{s+4\pi^2}\sin 2\pi x$
- Determine the values of c1 and c2. Taking the Laplace transform of those boundary conditions that involve t, we obtain c1 =0, c2 = 0. Thus (2) becomes $u = \frac{3}{s+4\pi^2}\sin 2\pi x$
- Inversion gives

  $$U(x,t) = 3e^{-4\pi^2 t}\sin 2\pi x$$

Solving Electrical Circuits Problem

Problem: From the theory of electrical circuits we know, $i = C\frac{dv}{dt}$

where C is the capacitance, i = i(t) is the electric current , and v = v(t) is the voltage.

We have to find the correct expression for the complex impedance of a capacitor.

Solution:

- Taking the Laplace transform of this equation, we obtain, $I(s) = C(sV(s) - V_o)$,

  Where, $I(s) = \mathcal{L}\{i(t)\}$, $V(s) = \mathcal{L}\{v(t)\}$, and $V_o = v(t)|_{t=0}$.

- Solving for V(s) we have $V(s) = \frac{I(s)}{sC} + \frac{V_o}{s}$.

- We know, $Z(s) = \left.\frac{V(s)}{I(s)}\right|_{V_o=0}$.

  So we find:

  $$Z(s) = \frac{1}{sC},$$

which is the correct expression for the complex impedance of a capacitor.

Other Application of Laplace Transform

- To determine structure of astronomical object from spectrum
- To find moment generating function in statistics
- To determine the present value of consol or bond in economics
- To solve the problem related to communication and network analysis.
- To make a equation in simple form from hard equation like vibration of spring.
- To solve Mixing Problem Involving Two Tanks

Limitation of Laplace Transform

Only be used to solve differential equations with known constants. An equation without the known constants, then this method is useless.

Conclusion

Laplace Transformation is powerful tool using in different areas of mathematics, physics and engineering.

With the ease of application of Laplace transforms in many applications, many research software have made it possible to simulate the Laplace transformable equations directly which has made a good advancement in the research field.


Thank you!